For the other containment, let $y \in f(A) \cup f(B)$. If $y \in f(A)$, then $y = f(x)$ for some $x \in A \subset A \cup B$. If $y \in f(B)$, then $y = f(x)$ for some $x \in B \subset A \cup B$. Thus, $y \in f(A \cup B)$.

2. By drawing the graph of f (this needs to be stressed to students), (a) through (d) should be clear.

   (a) $\{-1\} \cup [0,1]$.

   (b) $(-\infty, 1]$.

   (c) $(-\infty, 0)$.

   (d) $\emptyset$.

   (e) $f^{-1}(f((-\infty,0))) = f^{-1}(\{-1\}) = (-\infty, 0)$.

   (f) $f(f^{-1}((-\infty,0))) = f((-\infty,0)) = \{-1\}$.

3. If $x \in f^{-1}(S \cap T)$, then $f(x) \in S \cap T$ and so $f(x) \in S$ and $f(x) \in T$. Therefore, $x \in f^{-1}(S)$ and $x \in f^{-1}(T)$ and so $x \in f^{-1}(S) \cap f^{-1}(T)$. For the other containment, reverse the above steps.

4. One easily shows that $A \subset f^{-1}(f(A))$ always [$x \in A \Rightarrow f(x) \in f(A) \Rightarrow x \in f^{-1}(f(A))$].
   Let $x \in f^{-1}(f(A))$. Then $f(x) \in f(A)$ and so $\exists\ a \in A$ such that $f(x) = f(a)$. Since f is 1-1, $x = a \in A$. Thus, $f^{-1}(f(A)) \subset A$ if f is 1-1.
   Example. Let $f(x) = x^2$ on $\mathbb{R}$. Let $A = [0,1]$. Then $f^{-1}(f(A)) = f^{-1}([0,1]) = [-1,1] \supsetneqq A$.

5. One easily shows that $f(f^{-1}(B)) \subset B$ always [$y \in f(f^{-1}(B)) \Rightarrow y = f(x)$ for some $x \in f^{-1}(B)$ and therefore $y = f(x) \in B$].
   Let $y \in B$. Since f is onto Y, $\exists\ x \in X$ with $f(x) = y$. So $f(x) \in B \Rightarrow x \in f^{-1}(B) \Rightarrow y = f(x) \in f(f^{-1}(B))$. Thus, $B \subset f(f^{-1}(B))$ if f is onto Y.
   Example. Let $f(x) = x^2$ on $\mathbb{R}$. Let $B = [-1,0]$. Then, $f(f^{-1}(B)) = f(\{0\}) = \{0\} \subsetneqq B$.

6.
   1. If $x_1 \neq x_2$ in X, then $f(x_1) \neq f(x_2)$ in Y since f is 1-1. Then $g(f(x_1)) \neq g(f(x_2))$ since g is 1-1.
   2. If $z \in Z$, since g is onto Z, $\exists\ y \in Y$ with $g(y) = z$. Since f is onto Y, $\exists\ x \in X$ with $f(x) = y$. Then $g \circ f(x) = z$.
   3. Combine 1. and 2.

7. For x_1 and x_2 in X, if $f(x_1) = f(x_2)$, then $g(f(x_1)) = g(f(x_2))$. Since $g \circ f$ is 1-1, $x_1 = x_2$. Therefore, f is 1-1.
   Example. Define f and g on $\mathbb{R}$ by $f(x) = e^x$ and $g(x) = x^2$. Then g is not 1-1 but $g \circ f(x) = e^{2x}$ is 1-1. Note that g is 1-1 on the range of f.

8. Let $z \in Z$. Since $g \circ f$ is onto Z, $\exists\, x \in X$ with $g \circ f(x) = z$. Letting $y = f(x) \in Y$, then $g(y) = z$ and so g is onto Z.
   Example. Define f and g on $\mathbb{R}$ by $f(x) = |x|$ and $g(x) = x^2$. Then f is not onto $\mathbb{R}$ = domain of g but $g \circ f(x) = x^2$ maps $\mathbb{R}$ onto $[0, \infty)$. Also, $f(x) = x^2$ can be used here.

9. From Example 1.11, $h \circ g(x) = \frac{1}{x+1}$ is a bijection of $(0, \infty)$ onto $(0, 1)$.

10. $f(x) = (b-a)x + a$ is the equation of the straight line segment joining $(0, a)$ to $(1, b)$. So f is a bijection of $(0, 1)$ onto (a, b). Taking the composition of f with $h \circ g$ of the previous exercise gives a bijection from $(0, \infty)$ onto (a, b).
    Note that f is also a 1-1 map from $[0, 1]$ onto $[a, b]$. This is a good place to ask students about a bijection from a closed interval onto an open interval, as mentioned in the introduction to this chapter.

11. Let $y_1, y_2 \in Y$ with $f^{-1}(y_1) = f^{-1}(y_2)$. By Proposition 1.13, $y_1 = f(f^{-1}(y_1)) = f(f^{-1}(y_2)) = y_2$. So f^{-1} is 1-1.
    If $x \in X$, then $f(x) \in Y$. By Proposition 1.13, $x = f^{-1}(f(x))$. So f^{-1} is onto X.

12. If $x \in (g \circ f)^{-1}(B)$, then $g \circ f(x) = g(f(x)) \in B$. So $f(x) \in g^{-1}(B)$ and $x \in f^{-1}(g^{-1}(B))$. Thus, $(g \circ f)^{-1}(B) \subset f^{-1}(g^{-1}(B))$. Reverse these steps for the other containment.

1.4 Mathematical Induction

The argument in our final Remark fails only in going from $p(1)$ is true to $p(2)$ is true because there is no overlap in the reduced sets.

Below we give the step in going from $p(k)$ is true to $p(k+1)$ is true.

1.

$$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}.$$

2.

$$\begin{aligned}
1^3+2^3+\cdots k^3+(k+1)^3 &= \left(\frac{k(k+1)}{2}\right)^2+(k+1)^3 \\
&= (k+1)^2\left(\frac{k^2}{4}+(k+1)\right) \\
&= (k+1)^2\frac{(k+2)^2}{4} \\
&= \left(\frac{(k+1)(k+2)}{2}\right)^2.
\end{aligned}$$

3. $1+3+\cdots(2k-1)+(2k+1)=k^2+(2k+1)=(k+1)^2$.

4. $2^{k+1}=2(2^k)<2(k!)\le(k+1)(k!)=(k+1)!$.

5.

$$\begin{aligned}
7^{k+1}-3^{k+1} &= (7^k\cdot 7-7\cdot 3^k)+(7\cdot 3^k-3^k\cdot 3) \\
&= 7(7^k-3^k)+4\cdot 3^k.
\end{aligned}$$

Since 4 divides 7^k-3^k by the induction hypothesis and 4 divides $4\cdot 3^k$, 4 divides $7^{k+1}-3^{k+1}$.

6. $(k+1)^5-(k+1)=(k^5-k)+5(k^4+2k^3+2k^2+k)$ and 5 divides each part.

7.

$$\begin{aligned}
\sum_{j=1}^{k+1}(-1)^{j+1}j^2 &= \sum_{j=1}^{k}(-1)^{j+1}j^2+(-1)^{k+2}(k+1)^2 \\
&= (-1)^{k+1}\sum_{j=1}^{k}j+(-1)^{k+2}(k+1)^2 \quad \text{(by ind. hyp.)} \\
&= (-1)^{k+1}\frac{k(k+1)}{2}+(-1)^{k+2}(k+1)^2 \quad \text{(by Ex. 1)} \\
&= \frac{(-1)^{k+1}(k+1)}{2}[k+(-1)2(k+1)] \\
&= (-1)^{k+2}\frac{(k+1)(k+2)}{2} \\
&= (-1)^{k+2}\sum_{j=1}^{k+1}j \quad \text{(by Ex. 1)}.
\end{aligned}$$

8. If $ka \leq x_1 + x_2 + \cdots + x_k \leq kb$ and $x_{k+1} \in [a, b]$, then

$$(k+1)a = ka + a \leq x_1 + x_2 + \cdots + x_k + x_{k+1} \leq kb + b = (k+1)b$$

and so

$$a \leq \frac{x_1 + x_2 + \cdots + x_k + x_{k+1}}{k+1} \leq b.$$

9. Let $X = \{x_1, \ldots, x_k, x_{k+1}\}$. Then $\{x_1, \ldots, x_k\}$ has 2^k subsets by the induction hypothesis. Since all other subsets of X are of the form $A \cup \{x_{k+1}\}$ where $A \subset \{x_1, \ldots, x_k\}$, there are 2^k subsets of X containing x_{k+1}. So the number of subsets of X is $2^k + 2^k = 2^{k+1}$.

10.

$$\begin{aligned}(1+x)^{k+1} &= (1+x)^k(1+x) \\ &\geq (1+kx)(1+x) \\ &= 1 + (k+1)x + kx^2 \\ &\geq 1 + (k+1)x.\end{aligned}$$

11.

$$\begin{aligned}(\cos t + i \sin t)^{k+1} &= (\cos t + i \sin t)^k (\cos t + i \sin t) \\ &= (\cos kt + i \sin kt)(\cos t + i \sin t) \\ &= (\cos kt \cos t - \sin kt \sin t) + i(\sin kt \cos t + \cos kt \sin t) \\ &= \cos(k+1)t + i \sin(k+1)t.\end{aligned}$$